Looking Back: an Ordered Network Model of Legal Precedent

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1 Introduction

How does law affect the decisions of a court? Explanations range from legal models [1] to attitudinal models[2] to strategic models.[3] However, the so-called strategic models assume attitudinal goals – policy goals – and add strategic interaction. Some recent work[4] uses game theory to describe endogenously how judges might have incentives to adhere more or less to precedent, retaining the policy orientation of attitudinal models but recognizing the influence of stare decisis.

At the same time, various approaches have been taken to measure networks of citations,[5] including citation of Supreme Court opinions. Most of the existing network measures were developed to analyze undirected or directed networks, which have been the subject of recent theoretical exploration.[6] Citation networks do not fit these models well because once an author makes a decision to cite or not cite another opinion it generally cannot be changed in response to how later opinions choose their citations. Time's arrow imposes an order on the nodes of the network of opinions, and this order affects strongly both the strategic game and the appropriate measures of the resulting network.

One promising approach by Fowler, Johnson, Spriggs, Jeon, and Wahlbeck (2007) borrows Internet search engine technology[7] to develop two measures of opinion importance, inward importance and outward importance. **Outward importance** describes how well grounded an opinion

[1] Brisbin (1996); Knight and Epstein (1996); Benesh and Reddick (2002)

[2] Segal and Spaeth (1993, 1996)

[3] Epstein, Knight, and Martin (2001); Schanzenbach and Tiller (2004); Hettinger, Lindquist, and Martinek (2004); Hammond, Bonneau, and Sheehan (2005)

[4] Bueno de Mesquita and Stephenson (2002)

[5] de Solla Price (1965); Glänzel and Schubert (1990); Smith (2005); Cross and Smith (2006)

[6] Dutta and Jackson (2000); Jackson and Watts (2002); Jackson (2005); Jackson and Rogers (2006)

[7] Kleinberg (1999)

is in precedent, and is proportional to the sum of the inward importance of the opinions it cites. Similarly, **inward importance** measures how much other opinions rely on it, and is proportional to the sum of the outward importance of the opinions citing it. This and some additional technical assumptions give two unique measures of the opinions in a network, one for inward importance and one for outward importance. These measures can change over time, providing dynamic measures of opinion importance. Fowler et al. (2007) show that each of these measures do better than other common measures of importance in predicting how frequently various courts cite specific Supreme Court decisions.

This paper makes two key additions to the existing literature. First, we propose a new kind of network structure, an ordered network, designed to capture the behavior of a citation network missed by models of undirected or directed networks. Second, we use outward and inward importance as utility functions and solve simplified versions of a citation game. The results suggest that the best way to be cited is to cite well: choosing the optimal cites to maximize current outward importance can maximize expected future outward and inward importance.

The result stated immediately above refers to results on inward importance. The computational results for the model were indeterminate on this point. This lack of support is likely to have been the result of a newly discovered inefficiency in the original implementation. I have started a run with a modified design that greatly increases efficiency (by a factor of 1000); this run should provide evidence one way or the other.

The paper proceeds first by defining an ordered network and relating it to directed and undirected networks. Next, we develop a citation network game, propose an equilibrium, and discuss why an analytic proof is elusive. Then we explain how we use computation methods to provide evidence that the proposed strategy is a Nash equilibrium, and report the results of this method. Finally, we discuss implications and directions for further research.

2 The Ordered Network Model

2.1 Definition

Let N be a set of n nodes corresponding to court opinions and let $A_{n\times n} \in \{0,1\}^{n^2}$ be an adjacency matrix of citations from the "row" node to the "column" node. The pair (N, A) defines a **directed network**, denoted $\mathcal{N}$. Let $t : N \to \mathbb{N}$ associate[8] a discrete time with each node $i \in N$, denoted $t(i)$ or t_i. We define an **ordered network** as a triple $\overrightarrow{\mathcal{N}} = (N, A, t)$ where $a_{ij} = 0$ for all $t_i < t_j$. If $t_i < t_j$ then we say i **precedes** j and j **succeeds** i. If $t_i = t_j$ then i and j are **contemporaries**. Given this terminology, an ordered network is a directed network in which nodes cannot link to successors. An ordered network is **inclusive** if nodes can link to contemporaries, and **exclusive** otherwise.

We assume that t is one-to-one, which means that "does not succeed" imposes a partial order[9] on the nodes. This lets us identify nodes with their associated time, abusing notation to write $i \in \mathbb{N}$. Without further loss of generality, identify N with $\{1, 2, \ldots, n\} \subset \mathbb{N}$ where the **natural ordering** t is the identity function.

2.2 Two Examples

Consider a docket of four opinions, $N = \{1, 2, 3, 4\}$ with the natural ordering $t(i) = i$ for all $i \in N$. Let $\overrightarrow{\mathcal{N}}_{\text{all}} = (N, A_{\text{all}}, t)$ and $\overrightarrow{\mathcal{N}}_2 = (N, A_2, t)$ define two ordered networks, where

$$A_{\text{all}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Given that the adjacency matrix of an ordered network is lower triangular with zeros on the diagonal, there are $2^6 = 64$ possible ordered citation networks on four opinions. $\overrightarrow{\mathcal{N}}_{\text{all}}$ represents a network in which each opinion cites all the previous opinions; $\overrightarrow{\mathcal{N}}_2$ represents a network in which each opinion cites the first opinion and the preceding opinion, as available.

[8]Where $\mathbb{N}$ is the set of natural numbers: $\{1, 2, 3 \ldots\}$.

[9]A **partial order** is a binary relation that is reflexive, antisymmetric, and transitive. Any function t into $\mathbb{N}$ makes "does not succeed" reflexive and transitive. Assuming t is one-to-one also makes "does not succeed" antisymmetric.

2.3 Ordered Networks Are Different

When t is one-to-one or the network is exclusive,[10] the structure of an ordered network dictates that between any two nodes there is exactly one possible link, either $i \rightarrow j$ or $j \rightarrow i$ but not both. Given this, it is natural to ask if an ordered network can be characterized as an undirected network. The two sets of networks are isomorphic; if A is the adjacency matrix for $\overrightarrow{\mathcal{N}}$, then $A + A^T$ is the (symmetric) adjacency matrix for the corresponding undirected network. However, important differences become evident when considering the behavior of nodes in forming and breaking links. For an undirected network, common assumptions are that a link is formed when both nodes agree and broken when either node decides to break it. For an ordered network, we assume that node formation decisions are unilateral, as with more general directed networks, and that they can only be formed by one of the two parties (the "newer" one.) The authors of opinion k do not need the consent of the authors of opinion $k-1$ in order to cite opinion $k-1$, so this is a reasonable setup.

3 Citation Network Game

There are three key tasks remaining in developing a citation network game. First, one must choose a sensible utility function. Second, one must capture the idea that one cannot always just cite all previous opinions. Third, not all opinions are equally appealing as citations.

3.1 Utility: Outward and Inward Importance

Kleinberg (1999) describes an iterative algorithm for identifying inwardly and outwardly important Web sites. Let x and y denote n-dimensional vectors of weights over the n nodes of a network, where x identifies the relative inward importance and y identifies the relative outward importance of each node. Start with uniform weights (all entries set to 1.) Update each x_i to be the sum of the y_j linking to it, and update each y_i to be the sum of the x_j to which it links. Divide each vector by its norm (length) so each has length 1. Repeat the updating until it converges. Kleinberg shows

[10] Note that if t is one-to-one then a node has no contemporaries other than itself; in this situation the network is exclusive exactly when nodes cannot link to themselves.

that this algorithm does converge and that it converges to the principal eigenvectors[11] of $A^T A$ and AA^T, denoted x^* and y^*, respectively.

Kleinberg uses these to rank Web sites, and so only uses the weights to order the nodes. Fowler et al. use percentile ranks – an ordinal measure – of Supreme Court decisions, not the weights themselves. But the weights of the nodes (coordinates of x^* and y^*) are uniquely identified up to a scalar multiple, so they form a ratio measure of relative weight. The question becomes, should this ratio-level information be kept or discarded in building a utility function? Instead of using percentile ranks, another function of the weights could be used for utility, such as $\log x^*, \log y^*$. Lacking clear intuition on this, we retain the raw weights as the utility but leave open the possibility that further empirical work may shed light on how best to use the information encoded in the weights.

Let S_i denote the opinions cited by opinion i. The inward and outward importance can change over time as other opinions choose citations, so utility has value that depends on the time evaluated. For opinion i and time t we denote the inward importance with $\text{in}_i(t|S_i)$ or just $\text{in}_i(t)$ and outward importance with $\text{out}_i(t|S_i)$ or $\text{out}_i(t)$. The analysis will proceed first using outward importance as the utility function, then proceed to the analysis using inward importance.

There is some question as to whether this algorithm captures or is at least equivalent to the empirical process generating useful notions of importance for legal opinions. The proportionality statements[12] seem intuitive, but they yield many solutions, one x^* for each eigenvector of $A^T A$ and one y^* for each eigenvector of AA^T. In other words, in a network of n opinions there are n choices of x^* and n choices of y^*. If the process by which one attaches importance to opinions corresponds to the Kleinberg's algorithm, then his choice of the principal eigenvector is the right one. Fowler et al. (2007) show that the choice of the principal eigenvectors does very well at predicting citation frequency by U.S. state courts, U.S. Courts of Appeals, and the U.S. Supreme Court. Both did better than inward and outward degree; inward and outward eigenvector centrality;[13] appearing on the Congressional Quarterly's Guide to the United States Supreme Court, the Oxford

[11] A principal eigenvector of a matrix is an eigenvector associated with the largest eigenvalue of the matrix. We inherit Kleinberg's assumption that the largest eigenvalue of the $A^T A$ appears once in the spectrum, which is reasonable for a large network.

[12] Outward importance is proportional to the sum of the inward importance of the opinions it cites; inward importance is proportional to the sum of the outward importance of the opinions citing it.

[13] Bonacich (1972)

Guide to Supreme Court Decisions, or the front page of the New York Times; or the amount of amici brief interest. In other circumstances, other eigenvectors might work well. Kleinberg finds there is important information encoded in some of the nonprincipal eigenvectors, so the use of these in describing opinion citations remains an open question.

Consider our examples[14] $\overrightarrow{\mathcal{N}}_{\text{all}}$ and $\overrightarrow{\mathcal{N}}_2$. The outward importance measures y^* after the last node has chosen are calculated by taking

$$A_{\text{all}}A_{\text{all}}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \qquad A_2A_2^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

and calculating the normalized (positive, length 1) principal eigenvectors,

$$y^*_{\text{all}} = \begin{bmatrix} 0.00 \\ 0.33 \\ 0.59 \\ 0.74 \end{bmatrix} \qquad y^*_2 = \begin{bmatrix} 0.00 \\ 0.46 \\ 0.63 \\ 0.63 \end{bmatrix}$$

Therefore, $\text{out}_1(4|\text{all}) = 0$, $\text{out}_2(4|\text{all}) = 0.33$, $\text{out}_3(4|\text{all}) = 0.59$, and $\text{out}_4(4|\text{all}) = 0.74$, while $\text{out}_1(4|A_2) = 0$, $\text{out}_2(4|A_2) = 0.46$, and $\text{out}_3(4|A_2) = \text{out}_4(4|A_2) = 0.63$. At time 3, we calculate these based on the top-left 3×3 submatrix, and find in both networks that $\text{out}_2(3) = 0.53$. In both networks, the outward importance of the second opinion decreases from $t = 3$ to $t = 4$. Inward importance x^* is calculated separately and similarly by examining the principal eigenvector of A^TA for each network.

3.2 Citation: Not All Opinions are Equally Relevant

Suppose $\overrightarrow{\mathcal{N}}$ is an exclusive ordered network of n opinions with the natural ordering. At each time $i \in N$ could we let opinion i choose to cite any of the prior opinions. However, real opinions cannot practically cite *any* previous opinion, just as real research papers cannot practically cite all or even each previous paper. Some opinions are easy to cite, others take a little more work

[14] These examples are intentionally chosen to be quite small to illustrate clearly the definitions and calculations. There are certain order-related phenomena which occur in this model that do not fit well real situations. For example, the first node must have an outward importance of zero and the last node is stuck with inward importance zero; budget constraints are not binding in the early part of the network; later nodes have more choices, which gives them a larger set to optimize over but increases their costs when optimizing. This game-theoretic analysis is concerned with the behavior of nodes "in the middle" of a large network, so we choose not to address directly concerns about being near the ends or with the changing size of the network. These measures are reasonable for static networks; however, statistical network measurement research might benefit from careful consideration of how changing network size bias these measures of importance.

to show how they relate, and still others are for all practical purposes ineligible for meaningful citation, whether because they are on completely unrelated topics, support the opposing view, or have not been read by the opinion author. To capture this,[15] we assume that each opinion i can choose to cite any opinion in R_i, the set of opinions **relevant** to i. Each prior opinion has fixed probability ρ of being relevant. For simplicity, we assume that each opinion has a fixed budget $B \in \mathbb{N}$ common across opinions to reflect the time and resource constraints of justices. Thus opinion i chooses $S_i \subseteq R_i$ such that $|S_i| \leq B$.

The game proceeds with each opinion choosing preceding opinions to cite subject to the budget constraint. The game ends when the last opinion is decided.

If the only goal of an opinion author is to maximize the importance of an opinion at the time written, then there is no "game" in the sense of involving *strategic interaction*. The model plays out as a straightforward step-by-step optimization: each author cites the previous opinions which maximize its outward importance at that time, without care for the future. However, this seems unlikely. Outward importance can change over time, so an author might try to anticipate how the opinions she cites might be cited by future opinion authors. Inward importance is always zero at the time an opinion is published. A model providing no reason to anticipate the choices of future opinion authors would provide no direct value for inward importance, which seems untenable given the empirical evidence that inward importance is a good predictor of when an opinion is cited. Therefore in this model, utility (outward or inward importance) is evaluated at the end of the game.[16]

3.3 A Proposed Equilibrium

Given that each author tries to anticipate the choices of future authors, there is room to consider possible equilibria. The empirical results of Fowler et al. suggest two behavioral hypotheses: au-

[15] This is perhaps the simplest way to impose a budget constraint. Of the many things abstracted away, the phenomenon most interesting to the authors is the notion that relevance of opinions is in some sense transitive. That is, if opinion a is relevant to opinion b and b is relevant to c, then a is somewhat relevant to c. For now, we limit our analysis to this simple constraint.

[16] Some alternate utility evaluation schemes are worth consideration, given the dynamic nature of the measures. Utility could be some combination of the two measures. Utility could be the sum of the utilities evaluated at each time since the opinion was decided, effectively integrating the measure over time. The measure could be integrated over the entire time of the game or over some fixed length interval after the opinion is decided, corresponding to the career of the opinion author. This model assigns utility to opinions, but utility could be assigned to justices who then can choose to cite themselves, etc.

thors try to cite opinions with high (1) inward or (2) outward importance. The direct connection between an opinion's outward importance and the inward importance of the opinions it cites suggests focusing on the latter. This leads to the following conjectures.

Conjecture 1. *Choosing previous relevant opinions with largest inward importance is a subgame-perfect Nash equilibrium under when utility is evaluated as* outward *importance.*

Conjecture 2. *Choosing previous relevant opinions with largest inward importance is a subgame-perfect Nash equilibrium under when utility is evaluated as* inward *importance.*

3.4 Problems with Finding an Analytic Proof

How might one prove these conjectures? The goal is to show that, if other opinions are citing the previous relevant opinions with the largest inward importance, then the current opinion can maximize the expected value of outward or inward importance at some future time by citing the opinions with the largest inward importance.

Direct analytic methods are not sufficient for this. The choice for each opinion depends on a large-scale eigenvector computation. Anticipating the actions of others requires integrating the result of an eigenvector problem for each future opinion over the distribution of which previous opinions might be relevant to each future opinion. The study of how perturbations in matrices affect the eigenvalues of a matrix is reasonably advanced (Stewart and Sun, 1990) but next to nothing is yet known about how the components of the first eigenvector are affected by perturbations. Making things more difficult, the matrix being perturbed is A, which is not the matrix whose principal eigenvector is of interest. Rather, $A^T A$ and AA^T are analyzed for their principal eigenvectors, which prevents direct application of extant matrix perturbation theory.

4 Methods

4.1 Computational Modeling: Caveats and Strengths

Analytic proof is elusive, but the goal remains. Another approach is to check many specific examples of networks to see if the proposed equilibrium works. This might seem at best overly convenient and at worst just plain wrong. However, if implemented with great care, a coherent argument can be made. With this in mind, some general principals of computational modeling are worth reviewing. To make inferences when using a computational approach,[17] one must be clear about

1. why a computational model is necessary,
2. what is being estimated rather than calculated analytically,
3. what specific assumptions (parameters and functional forms) are being added to complete the computations,
4. which results imply "success," which imply "failure," and which are indeterminate,
5. which values of the parameters/functional forms specified in (3) give the best chance of falsifying the desired result, and afterward
6. what are the limitations of the results.

By following this approach, one can make a compelling argument using computation, but there are some limitations. When making inferences about situations that fall between vectors of tested parameters, the results are compelling only as long as the researcher has good intuition about how the phenomenon of interest changes as the parameters change. This is best accomplished by having substantive intuition, examining simple opinions analytically, and exploring a variety of more complex examples computationally. With this, the researcher can focus on making the test of the claims as tough as possible; without this, the researcher can fall into the trap of finding the rare instance in which the claims hold.

There are many strengths to computational methods when used in conjunction with other approaches; at least two apply to this study. One can investigate models that have fewer assumptions than formal models, assumptions that are usually made for analytic tractability. Everything is observable, so there are no latent variables, and as much "data" as desired can be generated, so the statistical tests can be much simpler.

Section 3.4 made our argument as to why a computational model is necessary. We continue by laying out parts (2) - (5) described above.

[17] de Marchi (2005); Miller and Page (2007)

4.2 Specifics: Estimates, Assumptions, Criteria for Success, and the Parameter Space

There are two quantities we estimate. First is the expected future value to the author of opinion i of choosing a set S_i of opinions to cite. Second, we cannot check all possible perturbations[18] from equilibrium behavior, so a random sample of possible perturbations is taken.[19]

To implement the model, we need to assume specific values for each of the parameters. For all runs of the model, we start with a single opinion with nothing to cite. Each time an opinion is added, the previous opinions are independently randomly determined to be relevant (with probability ρ) or not. The inward importance of the relevant opinions[20] is determined. The current opinions cites the most inwardly important relevant opinions up to the budget constraint.[21] This continues until there are 199 opinions in the network. This set of 199 first opinions is called a **start**.

Opinion 200 is the opinion being examined for equilibrium behavior. For each start, we calculate the set of relevant opinions and inward importance scores. We choose the equilibrium citations as before, and also choose 100 other sets of citations to be considered as perturbations from the equilibrium path. Each perturbation is a random sample without replacement from the set of relevant previous opinions. Each of these $1 + 100 = 101$ possibilities is called a **path**.

Opinions 201-400 choose their citations the following the same behavior as opinions 1-199, with possibly different choices for each of the 101 paths. These future choices are completely deterministic (constant for a given path) except for two factors: which previous opinions are relevant to each future opinion, and how are ties broken. Ties are very unlikely, so the almost all of the uncertainty comes from relevance. For each path we want $E\left[\mathrm{in}_{200}(400)\right]$ and $E\left[\mathrm{out}_{200}(400)\right]$, the expected future value of the importance measures of opinion 200, where expectation is with respect to these sources of uncertainty.

A **set of future relevances** is a list of opinions relevant to each opinion 201-400. it includes a subset of opinions 1-200 for opinion 201, a subset of opinions 1-201 for opinion 202,. . . , and a subset

[18]Suppose there are 20 relevant previous opinions and the budget is 10. Then there are $\binom{20}{10} = 184,756$ possible sets of opinions that can be cited.

[19]We wish to show that the proposed equilibrium strategy fares better than any perturbation. Instead of just taking a random sample, one might also include "likely suspects" such as sets of citations that have a lot of opinions in common with the proposed equilibrium set. These results are preliminary; future runs will include an oversampling of sets "close to" the proposed equilibrium set.

[20]Inward and outward importance are defined for the entire matrix of previous opinions, and must be calculated simultaneously for relevant and irrelevant opinions, but only the relevant opinions can be cited.

[21]Sometimes the budget constraint is slack because the number of relevant opinions is less than the budget. In this situation, the current opinion cites all of the relevant opinions. Ties in importance are broken randomly.

of opinions 1-399 for opinion 400. For each set of future relevances we calculate $\text{in}_{200}(400)$ and $\text{out}_{200}(400)$ once for each path by letting each opinion 201-400 choose their citations. Repeating across 500 sets of future relevances and taking a mean across those future relevances gives us $E\left[\text{in}_{200}(400)\right]$ and $E\left[\text{out}_{200}(400)\right]$.

To refute the conjecture, we must find a situation where the proposed equilibrium fares less well than some perturbation. We evaluate this with a one-tailed paired t-test of the 500 observed values of $\text{in}_{200}(400)$ and $\text{out}_{200}(400)$, comparing each perturbation with the proposed equilibrium.

To increase the robustness of the results, we run the model for budget values in $\{10, 20\}$, $\rho \in \{0.1, 0.2, 0.5\}$, and across five independent starts. This means that we run the model

$$(2 \text{ budgets}) \times (3 \text{ values of } \rho) \times (5 \text{ starts}) \times (101 \text{ paths}) \times (500 \text{ futures}) = 1,515,000 \text{ runs}$$

4.3 Implementation

Each run takes about half a minute on a modern desktop, so this calculation can be expected to take a little over 17 months. However, each 30 second run can be computed at the same time as every other run, so the algorithm parallelizes simply.

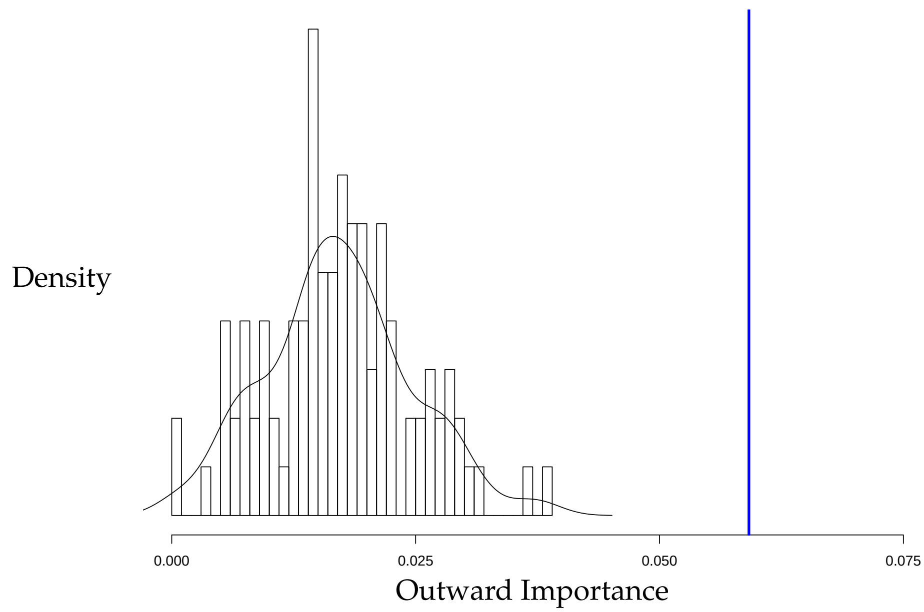
We break up the work into stages. First, we calculate the five starts (citations of opinions 1-199,) the proposed equilibrium and 100 perturbations[22] for opinion 200, and 500 sets of future relevances.[23] These are stored on a Web server. An `R` script downloads a work unit from the Web server, completes the run for opinions 201-400, then uploads the results to a database. This methods allows us to complete the calculations in less than two weeks using almost any collection of Internet-connected computers.[24]

[22]Even random choices for opinion 200 depend on the relevance of the previous opinions, so each start must have different perturbations.

[23]Relevances are assumed to be independent, so they are not path-dependent. This allow us to use the same 500 sets of future relevances for all five starts.

[24]Complete details and `R/PHP/MySQL` code are available at `http://haptonstahl.org/srh/`.

Figure 1: Outward Importance for Opinion 200 Given Equilibrium and Perturbed Citation Sets for Start $= 1$, $\rho = 0.1$, and Budget $= 10$



Notes: The dark vertical line on the right marks the expected outward importance after opinion 400 given that opinion 200 cited the 10 relevant previous opinions with the largest inward importance. The histogram and kernel density plots show expected outward importance after opinion 400 for the 100 perturbed citation choices by opinion 200. Expected values for each of the 101 paths calculated by taking the mean over about 1007 runs each (minimum number of runs > 900 per path.)

5 Results

The results here are preliminary, based on an inefficient algorithm. Specifically, for each start I specified one set of future relevances, and held that constant over the ≈ 1000 runs for each path. This means that I "integrated" out the uncertainty about how ties are broken, but I checked only one set of future relevances per start. This is different from the design described above. I am currently running the model using the correct setup described above; I expect the final results to provide much stronger support for both conjectures.

Figure 1 is an example of the results. In this figure there are two key pieces of information. The heavy vertical line shows the expected future outward importance of opinion 200 given behavior on the conjectured equilibrium path. The histogram and kernel density plot show the distribution of expected future outward importance given the 100 perturbations from the proposed equilibrium. This is a typical example, in that the conjectured equilibrium is far to the right, and thus fares much better than the evaluated perturbations.

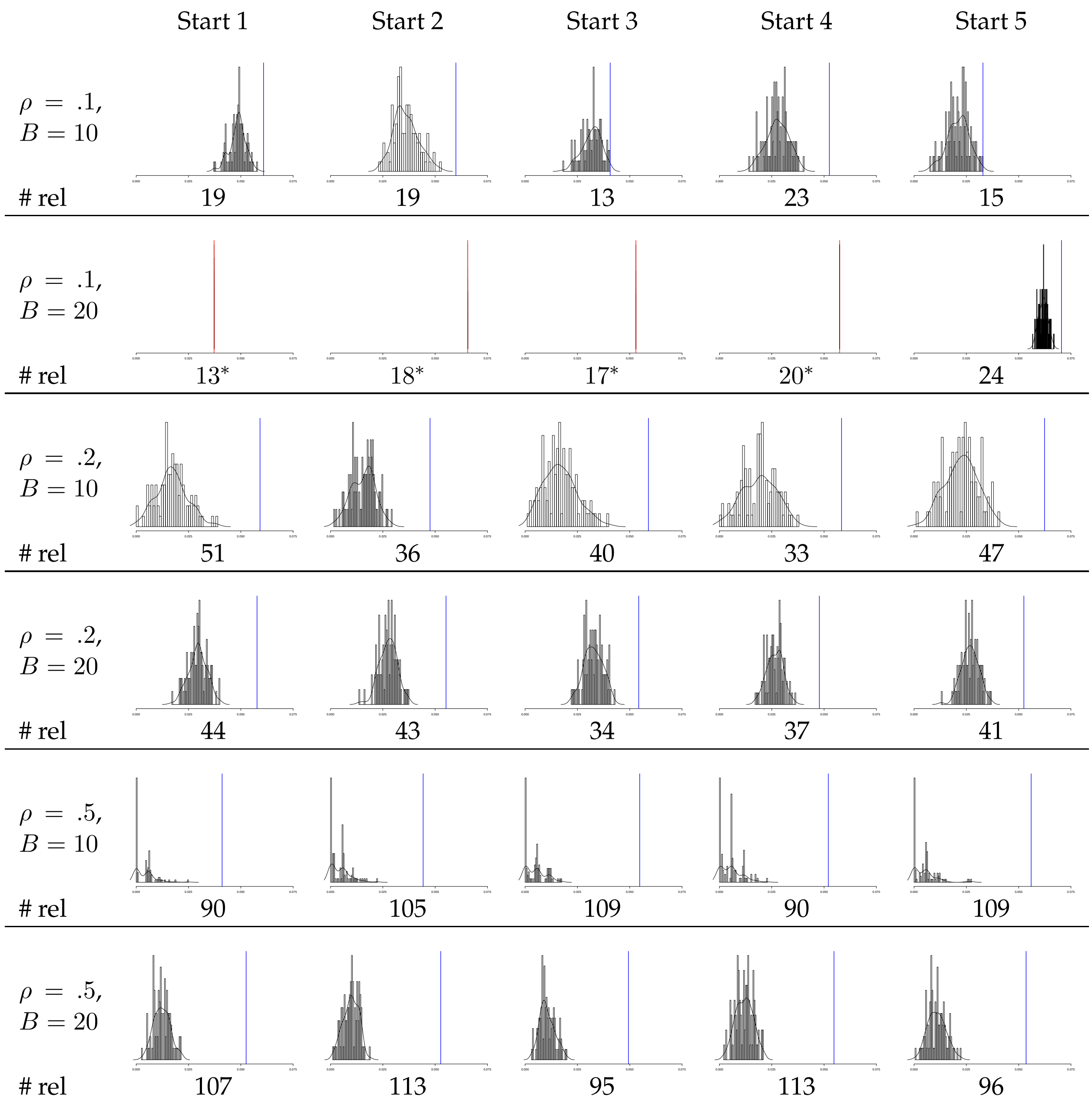
Figure 2 shows all of the results for outward importance. Most of the plots show strong support for Conjecture 1. The plots that appear to provide little support for the conjecture actually provide little information and no counter-evidence.

The first four plots in the second row seem problematic because they do not show any variation: all of the paths appear to have the same expected future outward importance. In these plots the number of previous opinions relevant to opinion 200 ("# rel") was less than or equal to the budget of 20, which means that all of the supposedly perturbed paths chose the same previous opinions as the equilibrium path. Therefore, it is not surprising that all of the paths in these plots had the same expected future outward importance.

Two other plots appear to provide only weak support for Conjecture 1: Start 3 and 5 in the first row. They do, in fact, support the hypothesis: a t-test comparing the runs of the proposed equilibrium path with the runs of any other path suggest that the mean of the proposed equilibrium utilities is statistically significantly greater ($p < 10^{-16}$). Still, the utilities of the perturbations are worryingly close to that of the proposed equilibrium. A check of the number of relevant previous cases provides a reasonable explanation. For row 1, start 3, there were only 13 relevant previous opinions and a budget of 10. This means that there were $\binom{13}{10} = 286$ possible sets of opinions that could be cited; there were only 286 possible "actions" that opinion 200 could take, and 100 of those 286 actions were evaluated. Furthermore, any action taken must have at least seven cited opinions in common with the proposed equilibrium. Citing the same opinions as the proposed equilibrium is highly correlated with expected future outward importance.[25] Similar reasoning applies to the plot in row 1 for start 5, so neither plot impeaches Conjecture 1.

[25]The correlation between fraction of cases in common with the proposed equilibrium strategy and expected future outward importance was calculated for each plot. The median correlation for the plots was 0.91; the smallest correlation was 0.53, which is still compelling.

Figure 2: Equilibrium and Perturbed Outward Importance for Opinion 200 at $t = 400$



Notes: The dark vertical line on the right marks the expected outward importance after opinion 400 given that opinion 200 cited the 10 relevant previous opinions with the largest inward importance. The histogram and kernel density plots show expected outward importance after opinion 400 for the 100 perturbed citation choices by opinion 200. Expected values for each of the 101 paths calculated by taking the mean over about 1007 runs each (minimum number of runs > 900 per path.) An asterisk (*) indicates that the budget constraint was slack because too few previous opinions were relevant.

The results for inward importance all looked like the problem situations in row 2 of Figure 2. The reason is that, for each start, only a single set of future relevant opinions was evaluated. This lack of variation meant that none of the opinions 201-400 cited opinion 200, giving the opinion of interest an inward importance of zero for all of the situations evaluated. This is a result of the mistaken algorithm; the corrected algorithm described in the text should prevent this and allow for testing of Conjecture 2.

6 Discussion

6.1 Implications

Even with these preliminary results, computational testing of the model clearly supports Conjecture 1, which states that citing the most inwardly importance previous relevant cases is a best response to the anticipated choices of future opinion authors if utility is the outward importance of a case. If an opinion author wants her opinion to be "well-grounded" in the existing network of opinion citations, citing inwardly important cases is a good strategy.

The preliminary results are indeterminate about the value of citing well if utility is the inward importance of a case. It is hoped that the current run using the improved configuration will provide evidence one way or the other.

If the results of the improved run strongly support both conjectures, what then? If the same strategy (choosing the previous relevant cases with largest current inward importance) maximizes two different utility functions (future inward and outward importance) then it maximizes any convex combination of those utilities. This means that a justice trying to balance writing a well-grounded opinion with writing an opinion likely to be cited faces no conflict: the same actions will optimize any weighted sum of these interests. Legal theories arguing that one of these concerns is more or less important than the other in influencing citation behavior become moot. The two motivations become observationally equivalent given solely the network of opinion citations.

6.2 Directions for Further Research

Having embraced a computational approach, a direct extension of this model would further generalize the enabling assumptions. For example, relevance here is represented as a discrete variable:

a previous case is relevant, or it is not. A more realistic approach would be to model relevance as taking values in $[0, 1]$ and to insist that relevance be correlated along a chain of opinions. This would eliminate the unrealistic situations permitted in the current model where opinion a is relevant to opinion b, b is relevant to c, but a is not relevant to c.

The principal eigenvector of the symmetrized adjacency matrices is used to gauge importance here as it is in Fowler et al. (2007), but Kleinberg (1999) does not stop there. The nonprincipal eigenvectors can have components of more than one sign, so this provides two sets of "important" (large absolute value) nodes, one with large positive components, one with large negative components. Kleinberg shows an example where a search for abortion related sites finds pro-life Web sites have second eigenvector components with one sign and pro-choice sites have components with the other sign. Future research might explore the connection between the valence of a citation (positive/negative) and other eigenvectors.

An ordered network is a directed network with some additional structure, the ordering function, t. However, inward and outward importance are defined for general directed networks which lack this ordering structure. Most if not all measures of network characteristics (degree, centrality, density, clustering, diameter) similarly assume a symmetry in the directed network that is broken in an ordered network. Further research should explore how to measure these ordered networks, either using current measures but drawing sensible inferences that do not assume temporal symmetry, or using new measures that weight appropriately for the order in the network.

Finally, the possibility exists that this technology might lead to methods for predicting which cases might be ripe for citation. If the overall framework proves useful, one might continue by exploring the mechanisms for determining the "relevance" of previous opinions to a given opinion, which ultimately could shed light on how justices create the network of precedent.

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